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Nonperturbative Renormalization of the Sine-Gordon/Coulomb Gas System for $\beta^2 < 8\pi$: A Functional Integral Approach

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A new method of analyzing the divergences of the sine-Gordon/Coulomb gas system is introduced. It is shown that for $\beta^2 < 8\pi$ all divergences may be eliminated by a nonperturbative renormalization of the ground-state energy.

KEY WORDS: Coulomb gas; sine-Gordon field; nonperturbative renormalization.

1. INTRODUCTION

The sine-Gordon/Coulomb gas (SG/CG) system has long been known[(1)] to possess ultraviolet divergences associated with contributions to the partition function coming from regions where charges of different signs occupy the same position. From the field-theoretic point of view, on the other hand, it was also observed[(2)] that for a certain range of the coupling constant (temperature), namely $4\pi \leqslant \beta^2 < 8\pi$, a careful treatment of the ultraviolet divergences which went beyond Coleman's renormalization was required in order to make the theory sensible.[(2)]

In a recent series of papers[(3,4)] a method was introduced in order to deal with the successive divergences which appear in the region $4\pi \leqslant \beta^2 < 8\pi$, connected to the coalescence of dipoles, quadrupoles, and so on. These authors showed in a nonperturbative way that all divergences of the SG/CG system could be eliminated by the introduction of constant counterterms. The basis of this method is the so-called multiscale decom-

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position of the scalar field[4] as well as the renormalization group methods developed in ref. 5.

In the present work, a new method of analyzing the ultraviolet divergences of the SG/CG system is introduced. The method is based on the choice of appropriate subdivisions of the integration regions in the various terms of the grand partition function and the subsequent analysis of the behavior of the integrand in each of them. This procedure makes it possible to obtain a general expression for the finite and divergent parts of each term of the grand partition function. The final step in our renormalization procedure involves a change in the summation order in the grand partition function which requires the consideration of the whole series on the fugacity. One then finds that the divergent part of the grand partition function factorizes and can be eliminated by a constant subtraction in the ground-state energy in agreement with ref. 3. The picture of ref. 3 that the constant to be subtracted from the ground-state energy contains the self-energies of the coalesced dipoles, quadrupoles, and so on, according to the value of β, also holds here. We believe, however, that our method is simpler and exhibits more directly the divergence structure of the theory. The basic idea behind our procedure was introduced in ref. 6 but only the dipolar singularities were considered there. We did not consider the problem of large-distance singularities. The stability of the system in the thermodynamic limit was studied in ref. 9.

In Section 2, we introduce our method of isolation of divergences, first when two and four particles, respectively, are present and then for the general case. Two Appendices are included to demonstrate results of this section. Section 3 is devoted to the actual renormalization process, whose key step is the nonperturbative change in summation order (3.1). Some conclusions and perspectives are presented in Section 4.

2. THE METHOD OF ISOLATION OF DIVERGENCES

2.1. The SG/CG Connection

Let us introduce in this section a method which will allow us to exhibit in a very clear way the complete divergence structure of the sine-Gordon/Coulomb gas system. The basic idea was introduced in ref. 6, but in that work, only a partial analysis of the divergences of the system was made.

We start with a very brief review of the connection between the sine-Gordon theory and the Coulomb gas of point particles,[7] which at the same time will set the basic formulas for later use.

The dynamics of the SG theory is determined by the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu \phi\, \partial^\mu \phi + 2\alpha_0 \cos \beta\phi \tag{2.1}$$

The CG version of the system may be obtained by making the expansion[6,7]

$$\exp\left(2\alpha \int d^2z \cos \beta\phi\right) = \sum_{n=0}^{\infty} \frac{\alpha_0^n}{n!} \sum_{\{\lambda_i\}_n} \int \prod_{i=1}^{n} d^2z_i \exp[i\beta\lambda_i \phi(z_i)] \tag{2.2}$$

in the vacuum functional $Z = Z_0^{-1} \int \mathscr{D}\phi \exp(-\int d^2z\, \mathscr{L})$. In the preceding expressions, $\lambda_i = \pm 1$ and $\sum_{\{\lambda_i\}_n}$ runs over all possibilities in the set $\{\lambda_1, ..., \lambda_n\}$. Here $Z_0 \equiv Z|_{\alpha_0 = 0}$ is the free vacuum functional. The large-distance behavior of the integrals may be regulated by putting the system inside a box of radius R, that is, by assuming $|z_i| < R$. Inserting (2.2) in the expression for Z, one may see that the functional integral becomes quadratic. This may be evaluated with the aid of the Euclidean scalar, massless Green function $D(z)$. In order to control the ultraviolet and infrared singularities, we introduce the regularized $D(z)$ as[6]

$$D(z) = \lim_{\varepsilon, \mu \to 0} -\frac{1}{4\pi} \ln \mu^2(|z|^2 + \varepsilon^2) \tag{2.3}$$

Performing the functional integration, one immediately gets[6]

$$Z = \lim_{\mu, \varepsilon \to 0} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{n!} \sum_{\{\lambda_i\}_n} \int \prod_{i=1}^{n} d^2z_i \exp\left\{\frac{\beta^2}{4\pi} \sum_{i \neq j}^{n} \lambda_i \lambda_j \ln(|z_i - z_j|^2 + \varepsilon^2) + \frac{\beta^2}{4\pi} \sum_{i=1}^{n} \ln \varepsilon^2 + \frac{\beta^2}{4\pi}\left(\sum_{i=1}^{n} \lambda_i\right)^2 \ln \mu^2\right\} \tag{2.4}$$

We see that Z is the grand-partition function of the two-dimensional Coulomb gas of point particles with charges $\pm\lambda_i$, with fugacity α_0, and $\beta^2/4\pi = 1/kT$.[6,7] The last term in the exponent of (2.4) forces the neutrality ($\sum_i \lambda_i = 0$) of the system in the limit $\mu \to 0$. The second term contains the first divergences we encounter: the self-energies of the n charges. These may be eliminated by a redefinition of the fugacity, namely, Coleman's renormalization[8]

$$\alpha = \alpha_0(\varepsilon^2)^{\beta^2/4\pi} \tag{2.5}$$

Due to the neutrality of the system, n must be even ($n = 2m$) and $\sum_{\{\lambda_i\}_n} = (2m)!/(m!)^2$, implying that the vacuum functional/grand-partition function will be given by

$$Z = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(n!)^2} \int \prod_{i=1}^{2n} d^2z_i \exp\left[\frac{\beta^2}{4\pi} \sum_{i \neq j}^{2n} \lambda_i \lambda_j \ln(|z_i - z_j|^2 + \varepsilon^2)\right] \tag{2.6a}$$

which we rewrite as

$$Z=\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(n!)^2}Z(n\mid n) \tag{2.6b}$$

$Z(n\mid n)$, which is defined by (2.6), is the partition function for a Coulomb gas with n positive and n negative point charges.

The integrand in (2.6) becomes singular, in the limit $\varepsilon\to 0$, in regions where one or more pairs of variables z_i and z_j become close to each other, for $\lambda_i\neq\lambda_j$, that is, when one or more pairs of charges of opposite sign coalesce. For $\beta^2<4\pi$ all these singularities are integrable and therefore expression (2.6) is finite in the limit $\varepsilon\to 0$. This was first observed in refs. 1 and 7. For these values of β, the only divergences of the theory are the ones associated with the self-energies of the charges and Coleman's renormalization (2.5) eliminates all of them. For $\beta^2\geqslant 4\pi$, however, the integrals in (2.6) start to diverge in regions where a single pair of charges of opposite sign coalesce. We therefore call these dipolar divergences.

As we increase the value of β, we start to have divergences associated to the coalescence of more and more pairs of charges of opposite sign. It is not difficult to find out at what value of β a given configuration of p charges of each sign coaslesced will start to diverge. This configuration, of course, may only occur in partition functions $Z(n\mid n)$ with $n\geqslant p$. Let us consider first the case $n=p$ with all the $2p$ charges coalesced inside an infinitesimal region of radius ε. This configuration, according to (2.6), will give a contribution to $Z(p\mid p)$ of the form

$$\begin{aligned}I_p(\varepsilon)&=\lim_{\varepsilon\to 0}V\int_{|z_i|<\varepsilon}\prod_{i=1}^{2p-1}d^2z_i\,\frac{(1,2)\cdots(p-1,p)(p+1,p+2)\cdots(2p-1,2p)}{(1,p+1)\cdots(p,2p)}\\&\equiv VJ_p(\varepsilon)\end{aligned} \tag{2.7}$$

where we defined $(i,j)\equiv(|z_i-z_j|^2+\varepsilon^2)^{\beta^2/4\pi}$ and chose $i=1,\ldots,p$ for the positive charges and $i=p+1,\ldots,2p$ for the negative ones. Observe that the overall volume factor may be interpreted as due to the integration over the position of the charge around which the others coalesced, $i=2p$ in the example above. Expression (2.7) possesses $2\cdot\binom{p}{2}(i,j)$ factors in the numerator and p^2 in the denominator. It is easy to see, by power counting, that in the limit $\varepsilon\to 0$, $I_p(\varepsilon)$ must behave as

$$I_p(\varepsilon)\equiv VJ_p(\varepsilon)\xrightarrow[\varepsilon\to 0]{}V\left(\frac{1}{\varepsilon^2}\right)^{(\beta^2/4\pi)p-(2p-1)} \tag{2.8}$$

We see, therefore, that a configuration containing p pairs of charges with opposite sign coalesced will diverge for $\beta^2\geqslant 4\pi(2p-1)/p$ [in the threshold, when the equality holds, the divergence of (2.8) will be logarithmic]. These thresholds are the ones found in ref. 3.

As we will see below, partition functions $Z(n|n)$ with $n>p$ will contain powers of the basic divergence $I_p(\varepsilon)$. These powers will express the number of times the configuration of $2p$ coalesced charges occurs. In this work, we are going to restrict the analysis to the case in which $\beta^2<8\pi$. For this range of values only the neutral coalesced configurations produce divergences. For $\beta^2 \geqslant 8\pi$ the charged configurations also start to diverge. We expect that this method can also be applied in this case. We are presently investigating this possibility.

2.2. The cases of $Z(1|1)$ and $Z(2|2)$

Let us introduce in this subsection the method of analysis of divergences in the two simplest cases, namely $n=1$ and $n=2$. In the next subsection we will consider the most general case.

Let us take first $Z(1|1)$, which is given by

$$Z(1|1)=\int_{\mathscr{R}} \frac{d^2 z_1\, d^2 z_2}{(1,2)} \tag{2.9}$$

where $\mathscr{R}$ is the integration region containing the variables z_1 and z_2 with $|z_1|$, $|z_2|<R$ and considered in the limit $R\to\infty$. Let us divide the integration region $\mathscr{R}$ in two parts, $\mathscr{D}_\delta$ and $\mathscr{F}_\delta$, in such a way that $|z_1-z_2|<\delta$ inside $\mathscr{D}_\delta$ with $\delta>0$ and $\mathscr{F}_\delta$ being the complement of $\mathscr{D}_\delta$ with respect to $\mathscr{R}(\mathscr{D}_\delta\cup\mathscr{F}_\delta=\mathscr{R}$ and $\mathscr{D}_\delta\cap\mathscr{F}_\delta=\varnothing)$. Of course, $\int_{\mathscr{R}}=\int_{\mathscr{D}_\delta}+\int_{\mathscr{F}_\delta}$ and therefore $Z(1,1)$ may be written

$$Z(1,1)=D(\varepsilon,\delta)+F(\varepsilon,\delta) \tag{2.10}$$

where $D(\varepsilon,\delta)$ and $F(\varepsilon,\delta)$ are the contributions from the regions $\mathscr{D}_\delta$ and $\mathscr{F}_\delta$, respectively. Since the divergence of $Z(1|1)$ comes from the region where the two charges are close together, $F(\varepsilon,\delta)$ must remain finite in the limit $\varepsilon\to 0$, while $D(\varepsilon,\delta)$ must diverge in the same limit. Observe that $Z(1|1)$ must be completely independent of δ $[(d/d\delta)(D+F)=0]$, which implies that D and F are respectively of the form

$$D(\varepsilon,\delta)=D(\varepsilon)+f(\delta,\varepsilon) \tag{2.11a}$$

$$F(\varepsilon,\delta)=F(\varepsilon)-f(\delta,\varepsilon) \tag{2.11b}$$

Notice that $f(\delta,\varepsilon)$ must depend on ε in such a way that $\lim_{\varepsilon\to 0} f(\delta,\varepsilon)\equiv f(\delta)=\text{finite}$ because otherwise $F(\varepsilon,\delta)$ would diverge in the limit $\varepsilon\to 0$, a fact which is impossible, as we saw. [Observe that in the limit

$\delta \to 0$, we must have $\mathscr{D}_\delta \to \varnothing$, meaning that $\lim_{\delta \to 0} D(\varepsilon, \delta) = 0$, that is, $\lim_{\delta \to 0} f(\delta, \varepsilon) = -D(\varepsilon)$.] The divergent part of $D(\varepsilon, \delta)$ in the limit $\varepsilon \to 0$ must, as a consequence of the finiteness of $f(\delta, \varepsilon)$, be contained in the $D(\varepsilon)$ term of (2.11a). Since the divergence of $D(\varepsilon)$ comes from the region where z_1 is close to z_2, we must have, according to the analysis made in the last subsection,

$$\lim_{\varepsilon \to 0} D(\varepsilon) = I_1(\varepsilon) \tag{2.12}$$

where $I_1(\varepsilon)$ is given by (2.7) and (2.8). On the other hand, $F(\varepsilon)$ must be finite in the limit $\varepsilon \to 0$.

Defining

$$\tilde{Z}(1 \mid 1) = \lim_{\varepsilon \to 0} F(\varepsilon) \tag{2.13}$$

we see, from (2.10)–(2.13), that in the limit $\varepsilon \to 0$

$$Z(1 \mid 1) = \tilde{Z}(1 \mid 1) + I_1(\varepsilon) \tag{2.14}$$

$\tilde{Z}(1 \mid 1)$ is the finite part of the partition function $Z(1 \mid 1)$. Observe that it is completely independent of the cutoff ε and also of the arbitrary parameter δ, which only serves to label the given partition of the integration region $\mathscr{R}$ we choose. $I_1(\varepsilon)$ is the divergent part of $Z(1 \mid 1)$.

In Appendix A we explicitly compute $Z(1 \mid 1)$ and obtain the expressions for $D(\varepsilon, \delta)$, $F(\varepsilon, \delta)$, $F(\varepsilon)$, $D(\varepsilon)$, $f(\delta, \varepsilon)$, $\tilde{Z}(1 \mid 1)$, and $I_1(\varepsilon)$.

Let us consider now the case of $Z(2 \mid 2)$, which is given by

$$Z(2 \mid 2) = \int_{\mathscr{R}} d^2z_1\, d^2z_2\, d^2z_3\, d^2z_4 \frac{(1, 2)(3, 4)}{(1, 3)(2, 4)(1, 4)(2, 3)} \tag{2.15}$$

The integral above will diverge due to the behavior of the integrand when the following configurations occur: (i) A single coalesced dipole. This may occur in the four subregions of $\mathscr{R}$ in which, respectively, $(z_1 - z_3)$, $(z_2 - z_4)$, $(z_1 - z_4)$, and $(z_2 - z_3)$ tend to zero separately. (ii) Two separately coalesced dipoles. This occurs in the two subregions where, respectively, $(z_1 - z_3)$ and $(z_2 - z_4)$ or $(z_1 - z_4)$ and $(z_2 - z_3)$ tend to zero. (iii) All four charges coalesced, that is, a coalesced quadrupole. This occurs in the region where $(z_i - z_j) \to 0$ for all values of $i, j = 1,..., 4$. [Observe that this divergence only appears for $\beta^2 \geqslant 6\pi$, according to (2.8). We are going to make our analysis for an arbitrary value of $\beta < 8\pi$.] Other potentially dangerous configurations would be the ones containing two charges of a given sign and a third one of the opposite sign. These nonneutral configurations, however, will only diverge for $\beta^2 \geqslant 8\pi$.

Inspired by the above considerations, we subdivide the integration region $\mathscr{R}$ into eight subregions $\mathscr{R}^i_\delta$, such that $\bigcup_{i=1}^8 \mathscr{R}^i_\delta = \mathscr{R}$ and $\mathscr{R}^i_\delta \cap \mathscr{R}^j_\delta = \varnothing$ for all values if i and j $(i \neq j)$. The regions with $i = 1,..., 4$ are defined, respectively, by $\{z_l \mid |z_1 - z_3 < \delta;\ |z_2 - z_4| > \delta;\ |z_2 - z_3| > \delta,\ |z_4 - z_1| > \delta,\ |z_2 - z_1| > \delta,\ |z_4 - z_3| > \delta\}$, $\{z_l \mid |z_2 - z_4| < \delta,\ |z_1 - z_3| > \delta,\ |z_1 - z_4| > \delta,\ |z_3 - z_2| > \delta,\ |z_3 - z_4| > \delta,\ |z_1 - z_2| > \delta\}$, $\{z_l \mid |z_1 - z_4| < \delta,\ |z_2 - z_3| > \delta,\ |z_2 - z_4| > \delta,\ |z_1 - z_2| > \delta,\ |z_3 - z_4| > \delta,\ |z_3 - z_1| > \delta\}$, and $\{z_l \mid |z_2 - z_3| < \delta,\ |z_1 - z_4| > \delta,\ |z_4 - z_2| > \delta,\ |z_1 - z_3| > \delta,\ |z_4 - z_3| > \delta,\ |z_1 - z_2| > \delta\}$. The ones with $i = 5, 6$ are defined, respectively, by $\{z_l \mid |z_1 - z_3| < \delta,\ |z_2 - z_4| < \delta,\ |z_2 - z_3| > \delta,\ |z_4 - z_1| > \delta,\ |z_2 - z_1| > \delta,\ |z_4 - z_3| > \delta\}$ and $\{z_l \mid |z_1 - z_4| < \delta,\ |z_2 - z_3| < \delta,\ |z_1 - z_3| > \delta,\ |z_2 - z_4| > \delta,\ |z_3 - z_4| > \delta,\ |z_2 - z_1| > \delta\}$. The region $\mathscr{R}^7_\delta$ is defined by $\{z_l \mid |z_i - z_j| < \delta, \forall_{i,j}\}$. Finally, $\mathscr{R}^8_\delta$ is the complement of $\bigcup_{i=1}^7 \mathscr{R}^i_\delta$ with respect to $\mathscr{R}$. In the expressions above, again, δ is an arbitrary positive parameter. Observe that the above-mentioned charged configurations which start to diverge at $\beta^2 = 8\pi$ occur in $\mathscr{R}^8_\delta$. As before, we are going to use the fact that

$$\int_{\mathscr{R}} = \sum_{i=1}^{8} \int_{\mathscr{R}^i_\delta}$$

The left-hand side is completely independent of δ. In Appendix B, we show that in the limit $\varepsilon \to 0$

$$\sum_{i=1}^{6} \int_{\mathscr{R}^i_\delta} = 4I_1(\varepsilon)\, \tilde{Z}(1 \mid 1) + 2I_1^2(\varepsilon) + 4f(\delta)\, \tilde{Z}(1 \mid 1) - 2f^2(\delta) + 4r(\delta) + 2s(\delta) \tag{2.16}$$

In this expression $I_1(\varepsilon)$, $\tilde{Z}(1 \mid 1)$, and $f(\delta)$ were defined above and $r(\delta)$ and $s(\delta)$ are unknown finite functions of δ. Let us call $Q(\varepsilon, \delta)$ and $G(\varepsilon, \delta)$, respectively, the integrals over $\mathscr{R}^7_\delta$ and $\mathscr{R}^8_\delta$. We may always write

$$\int_{\mathscr{R}^7_\delta} \equiv Q(\varepsilon, \delta) = Q(\varepsilon) + q(\varepsilon, \delta) \tag{2.17a}$$

$$\int_{\mathscr{R}^8_\delta} \equiv G(\varepsilon, \delta) = G(\varepsilon) - g(\varepsilon, \delta) \tag{2.17b}$$

Observe first that $G(\varepsilon, \delta)$ must be finite in the limit $\varepsilon \to 0$, because the integrand of (2.15) is always regular in $\mathscr{R}^8_\delta$ for $\beta < 8\pi$. Finiteness is intrinsic and not due to cancelation of divergences. As a consequence, $\lim_{\varepsilon \to 0} g(\varepsilon, \delta) \equiv g(\delta)$ must be finite. Now, since the whole integral

$$\sum_{i=1}^{8} \int_{\mathscr{R}^i_\delta}$$

must be independent of δ, all δ-dependent terms must cancel, namely,

$$g(\delta)=q(\delta)+4f(\delta)\,\tilde{Z}(1\,|\,1)-2f^2(\delta)+4r(\delta)+2s(\delta) \tag{2.18}$$

Since $g(\delta)$, $f(\delta)$, $r(\delta)$, and $s(\delta)$ are finite, $q(\delta)$ must also be finite, implying that the divergent part of $Q(\varepsilon, \delta)$ in the limit $\varepsilon \to 0$ must be in the $Q(\varepsilon)$ term. According to the analysis made in Section 2.1, we must have

$$\lim_{\varepsilon \to 0} Q(\varepsilon)=I_2(\varepsilon) \tag{2.19}$$

For the reason stated above, $\lim_{\varepsilon \to 0} G(\varepsilon, \delta)=$ finite. Defining

$$\tilde{Z}(2\,|\,2)\equiv \lim_{\varepsilon \to 0} G(\varepsilon) \tag{2.20}$$

we may write, using (2.16)–(2.20),

$$Z(2\,|\,2)=4I_1(\varepsilon)\,\tilde{Z}(1\,|\,1)+2I_1^2(\varepsilon)+I_2(\varepsilon)+\tilde{Z}(2\,|\,2) \tag{2.21}$$

$\tilde{Z}(2\,|\,2)$ is the finite part of $Z(2\,|\,2)$. Observe that it is completely independent of ε and δ.

Expression (2.21) clearly exhibits the divergence structure of $Z(2\,|\,2)$ separating the various divergent parts from the finite one.

2.3. General Case

Let us obtain now an expression which generalizes (2.14) and (2.21) for an arbitrary partition function $Z(n\,|\,n)$. The method described above, which allows the exhibition of the divergence structure of $Z(n\,|\,n)$, is based on the identification of the various possible configurations of charges occupying the same place, which will give a divergent contribution to $Z(n\,|\,n)$. For $\beta^2<8\pi$, these are neutral configurations of p charges of each sign occurring one or more times, according to whether $p=n$ or $p<n$. The remaining charges not appearing in the coalesced configuration give a multiplicative finite factor $\tilde{Z}(m\,|\,m)$ with $m<n$, as in the first term of (2.21). After identifying the divergent configurations, by sweeping all possible values of p, one has to find the correct numerical factors which multiply each power of the basic divergent contributions.

Let us consider first the divergences produced by the coalescence of a configuration with a fixed p in the partition function $Z(n\,|\,n)$, with both p and n arbitrary. A series of arguments of the type presented in Section 2.2 plus a simple combinatoric analysis shows that

$$Z(n|n)=\sum_{l=0}^{[n/p]}\binom{n}{p}\binom{n}{p}\cdot\binom{n-p}{p}\binom{n-p}{p}$$
$$\times\cdots\binom{n-(l-1)p}{p}\binom{n-(l-1)p}{p}\frac{I_p^l(\varepsilon)}{l!}\tilde{Z}_p(n-lp|n-lp)$$

or

$$Z(n|n)=\sum_{l=0}^{[n/p]}\frac{(n!)^2 I_p^l(\varepsilon)}{(p!)^{2l}\,l!\,[(n-lp)!]^2}\tilde{Z}_p(n-lp|n-lp) \tag{2.22}$$

In the expression above, $[n/p]$ is the integer part of n/p, $I_p(\varepsilon)$ is the basic divergence produced by the configuration with a fixed p, and $\tilde{Z}_p(m|m)$ is the part of $Z(m|m)$ which is free of these divergences $[Z_p(0|0)\equiv 1]$. Observe that $l=0$ for $p>n$.

Expression (2.22), however, does not exhibit the complete divergence structure of $Z(n|n)$, because $Z_q(m|m)$ still may contain divergences associated with configurations with $p\neq q$. It is not difficult to obtain the expression which generalizes (2.22) and takes into account all values of p from $p=1$ to $p=N\to\infty$. A further combinatoric analysis gives

$$Z(n|n)=\lim_{N\to\infty}\sum_{l_1=0}^{[n/1]}\cdots\sum_{l_p=0}^{\left[\left(n-\sum_{q=1}^{p-1}l_q q\right)/p\right]}\cdots\sum_{l_N=0}^{\left[\left(n-\sum_{q=1}^{N-1}l_q q\right)/N\right]}$$
$$\times\frac{(n!)^2}{[(n-\sum_{p=1}^{N}l_p p)!]^2}\left\{\frac{I_1^{l_1}(\varepsilon)}{l_1!\,(1!)^{2l_1}}\right\}\cdots\left\{\frac{I_N^{l_N}(\varepsilon)}{l_N!\,(N!)^{2l_N}}\right\}$$
$$\times\tilde{Z}\left(n-\sum_{p=1}^{N}l_p p\,\Big|\,n-\sum_{p=1}^{N}l_p p\right) \tag{2.23}$$

In this expression, $\tilde{Z}(m|m)$ is the completely finite part of $Z(m|m)$, which was defined in Section 2.2 for the cases $m=1, 2$. It is completely independent of ε and δ. As before, $\tilde{Z}(0|0)\equiv 1$. Observe that $l_p=0$ whenever $n-\sum_{q=1}^{p-1}l_q q<p$.

Expression (2.23) displays the complete divergence structure of $Z(n|n)$. Its finite part, $\tilde{Z}(n|n)$, is given by the term with all $l_i=0$, for $i=1,\ldots,\infty$. That expression completes our analysis of the divergences of the partition functions $Z(n|n)$. In Ref. 6 we only considered the case $p=1$ and obtained a particular case of (2.23).

3. RENORMALIZATION: EXTRACTION OF THE MULTIPOLAR DIVERGENCES

Let us show here how we may eliminate in a nonperturbative way the divergences of the grand-partition function (2.6) which were uncovered in the last section.

We start by introducing (2.23) in (2.6). Then, a key step in our renormalization procedure follows: the realization that we may reorder the summations in (2.6) and (2.23) in the following way:

$$\sum_{n=0}^{\infty} \sum_{l_1=0}^{[n/1]} \cdots \sum_{l_N=0}^{\left[\left(n-\sum_{q=1}^{N-1} l_q q\right)/N\right]} = \sum_{l_1=0}^{\infty} \cdots \sum_{l_N=0}^{\infty} \sum_{n=\sum_{q=1}^{N} l_q q}^{\infty} \tag{3.1}$$

This reordering, of course, is based on the assumption that the series (2.6) with $Z(n\,|\,n)$ given by (2.23) exists before the cutoff ε is removed or, in other words, that the expansion (2.2) makes sense. It is not difficult to realize that the summations in both sides of (3.1) sweep the same values of n and l_i, $i=1,..., N$. Performing the change of summation variable $m=n-\sum_{q=1}^{N} l_q q$ in the last sum on the right hand side of (3.1), we get, taking into account (2.6), (2.23), and (3.1),

$$Z=\prod_{p=1}^{\infty} \left\{ \sum_{l_p=0}^{\infty} \frac{\alpha^{2pl_p} I_p^{l_p}(\varepsilon)}{(p!)^{2l_p}\, l_p!} \right\} \sum_{m=0}^{\infty} \frac{\alpha^{2m}}{(m!)^2} \tilde{Z}(m\,|\,m) \tag{3.2a}$$

or

$$Z=Z(\varepsilon)\, \tilde{Z} \tag{3.2b}$$

We see that all divergences of Z factor out in $Z(\varepsilon)$. The finite part of the grand-partition function/vacuum functional Z is given by the second term of (3.2), namely

$$\tilde{Z}=\sum_{m} \frac{\alpha^{2m}}{(m!)^2} \tilde{Z}(m\,|\,m)$$

It is made of the completely finite terms $\tilde{Z}(m\,|\,m)$ and therefore is itself free of ultraviolet divergences in the limit $\varepsilon \to 0$.

The divergent part of Z may be written as

$$Z(\varepsilon)=\prod_{p=1}^{\infty} \exp\left\{\frac{\alpha^{2p}}{(p!)^2} I_p(\varepsilon)\right\} = \exp\left\{\sum_{p=1}^{\infty} \frac{\alpha^{2p}}{(p!)^2} I_p(\varepsilon)\right\} \tag{3.3}$$

where we performed the sum in l_p in (3.2a). Taking (2.8) in account, we may write (3.3) as $Z(\varepsilon) = e^{VW(\varepsilon)}$, where

$$W(\varepsilon) = \sum_{p=1}^{\infty} \frac{\alpha^{2p}}{(p!)^2} J_p(\varepsilon)$$

Using (2.8), we may obtain a rough closed expression for $W(\varepsilon)$ (valid outside the thresholds), namely

$$W(\varepsilon) \sim \left(\frac{1}{\varepsilon^2}\right) \{I_0[2\alpha(\varepsilon^2)^{1-\beta^2/8\pi}] - 1\} \tag{3.4}$$

where I_0 is a modified Bessel function. In obtaining (3.4), we neglected the finite multiplicative factors which multiply (2.8).

The renormalized grand-partition function/vacuum functional Z_R is simply given by

$$Z_R \equiv Z^{-1}(\varepsilon) Z = e^{-VW(\varepsilon)} Z = \sum_{m=0}^{\infty} \frac{\alpha^{2m}}{(m!)^2} \tilde{Z}(m \mid m) \tag{3.5}$$

Taking into account the functional integral representation of Z, $Z = Z_0^{-1} \int D\phi \exp(i \int d^2 z \mathscr{L})$, and the fact that $V = \int d^2 z_E \to i \int d^2 z$, we may see that the renormalization procedure above is equivalent to a subtractive renormalization of the vacuum energy of the theory, namely

$$\mathscr{L}_0 \to \mathscr{L}_0 - W(\varepsilon) \tag{3.6}$$

We see that this renormalization consists in the extraction of the infinite series of multipolar divergences from the vacuum energy, in agreement with the results of ref. 3.

We would like to stress that the renormalization process employed here is nonperturbative, since the factorization of divergences depends crucially on the reordering of summations (3.1) which is only possible when the whole series in α is considered.

In ref. 6 a similar factorization of divergences was obtained, but only the case $p = 1$ was considered.

4. CONCLUSIONS AND PERSPECTIVES

We considered the SG/CG system and made a detailed analysis of its divergences employing a new method of analyzing the divergences, which separates the integration regions where certain multipolar configuration of charges give singular contributions to the integrals appearing in the expansion of the grand-partition function/vacuum functional of the theory. The remaining finite parts are completely cutoff independent. The whole set of divergences is shown to be eliminated by a subtraction in the vacuum energy. This renormalization is nonperturbative in the sense that the whole series in the fugacity must be considered before the subtraction may be performed. Our results agree with those of refs. 3 and 4 but we believe our method is simpler.

We are trying to extend the method for the application in the region where $\beta^2 \geqslant 8\pi$. In this region the theory is perturbatively unrenormalizable and therefore it would be extremely interesting to see whether the method works in this case. We also considered the supersymmetric sine-Gordon theory, where the application of our method produces very interesting results.[10]

APPENDIX A

Let us compute here the partition function $Z(1|1)$ given by (2.9). Making the change of variable $z = z_1 - z_2$ in the $R \to \infty$ limit, we get

$$Z(1|1) = V \int \frac{d^2 z}{(|z|^2 + \varepsilon^2)^{\beta^2/4\pi}} = 2\pi V \int_0^R \frac{r\,dr}{(r^2 + \varepsilon^2)^{\beta^2/4\pi}} \tag{A.1}$$

where V is the volume of the system and $R \to \infty$. The integration regions $\mathscr{D}_\delta$ and $\mathscr{F}_\delta$ introduced in Section 2 are defined by $r \equiv |z| < \delta$ and $r \equiv |z| > \delta$, respectively. Writing $\int_0^R = \int_0^\delta + \int_\delta^R$ and making the changes of variable $u = r^2$ and $v = u + \varepsilon^2$, one easily finds

$$Z(1|1) = \left\{ \frac{\pi V}{1 - \beta^2/4\pi} \left[(\delta^2 + \varepsilon^2)^{1 - \beta^2/4\pi} - (\varepsilon^2)^{1 - \beta^2/4\pi} \right] \right\}$$
$$+ \left\{ \frac{\pi V}{1 - \beta^2/4\pi} \left[(R^2 + \varepsilon^2)^{1 - \beta^2/4\pi} - (\delta^2 + \varepsilon^2)^{1 - \beta^2/4\pi} \right] \right\} \quad \text{for } \beta^2 > 4\pi \tag{A.2a}$$

or

$$Z(1|1) = \{ \pi V [\ln(\delta^2 + \varepsilon^2) - \ln \varepsilon^2] \}$$
$$+ \{ \pi V [\ln(R^2 + \varepsilon^2) - \ln(\delta^2 + \varepsilon^2)] \} \qquad \text{for} \quad \beta^2 = 4\pi \tag{A.2b}$$

In the expression above, the first term between curly brackets is $D(\varepsilon, \delta) \equiv f(\delta, \varepsilon) + D(\varepsilon)$. The second term between curly brackets is $F(\varepsilon, \delta) \equiv F(\varepsilon) - f(\delta, \varepsilon)$. Observe that $f(\delta, \varepsilon)$ is finite and disappears from the expression of $Z(1|1)$. Note also that $D(\varepsilon)$ is the divergent part of $D(\varepsilon, \delta)$ in the limit $\varepsilon \to 0$ and behaves as (2.8) in this limit. Notice that $D(\varepsilon, \delta) \to 0$ in the limit $\delta \to 0$. This was to be expected since $\mathscr{D}_\delta \to \varnothing$ in this limit. Observe, finally, that $F(\varepsilon)$ is finite in the limit $\varepsilon \to 0$ and the finite part of the partition function $Z(1|1)$ is given by ($V = \pi R^2$)

$$\tilde{Z}(1|1) \equiv \lim_{\varepsilon \to 0} F(\varepsilon) = \begin{cases} \dfrac{\pi V}{1 - \beta^2/4\pi} \left(\dfrac{V}{\pi} \right)^{1 - \beta^2/4\pi} & \beta^2 > 4\pi \\ \pi V \ln \dfrac{V}{\pi} & \beta^2 = 4\pi \end{cases} \tag{A.3}$$

APPENDIX B

Let us demonstrate here expression (2.16). Let us call C_i the contribution from the region $\mathscr{R}^i_\delta$,

$$C_i = \int_{\mathscr{R}^i_\delta} d^2z_1\, d^2z_2\, d^2z_3\, d^2z_4 \frac{(1,2)(3,4)}{(1,3)(2,4)(1,4)(2,3)} \tag{B.1}$$

and consider first the case $i=1$, remembering that $\mathscr{R}^i_\delta = \{z_i \mid |z_1 - z_3| < \delta, |z_2 - z_4| > \delta, |z_2 - z_3|, |z_1 - z_4|, |z_1 - z_2|, |z_3 - z_4| > \delta\}$. Let us take the piece $(1,2)(3,4)/(1,4)(3,2) \equiv f(1,2,3,4)$ of the integrand in (B.1). Observe that $f(1,2,3,4)$ and all its derivatives with respect to each variable z_i, $i = 1,..., 4$, are analytic in $\mathscr{R}^1_\delta$. We therefore expand z_3 around z_1 in $f(1,2,3,4)$, getting

$$f(1,2,3,4) = 1 + (z_3 - z_1)^\mu \left.\frac{\partial f}{\partial z_3^\mu}\right|_{z_3 = z_1} + \frac{1}{2}(z_3 - z_1)^\mu (z_3 - z_1)^\nu \left.\frac{\partial^2 f}{\partial z_3^\mu\, \partial z_3^\nu}\right|_{z_3 = z_1} + \cdots \tag{B.2}$$

Introducing (B.2) in (B.1), we see that the first term is nothing but $D(\varepsilon, \delta)$ $F(\varepsilon, \delta)$ (introduced in Section 2.2). Making the change of variable $x \equiv z_3 - z_1$ in the second term, one easily sees that the angular integration in d^2x forces it to vanish. The rest of the terms corresponding to the expansion (B.2) are finite because the integrands are either finite or contain integrable singularities for $\beta^2 < 8\pi$ in the region $\mathscr{R}^1_\delta$. We call $r(\varepsilon, \delta)$ the part of C_1 which corresponds to all terms in (B.2) starting with the third one. As we explained, this must be finite in the limit $\varepsilon \to 0$ because the ultraviolet singularities are either finite or integrable. We call $r(\delta) \equiv \lim_{\varepsilon \to 0} r(\varepsilon, \delta)$. To summarize, we have $C_1 = D(\varepsilon, \delta)\, F(\varepsilon, \delta) + r(\delta)$ in the limit $\varepsilon \to 0$. One can easily show that $C_1 = C_2 = C_3 = C_4$, by just permuting the integration variables (e.g., $C_1 = C_2$ by making the changes of variable $z_1 \leftrightarrow z_2$, $z_3 \leftrightarrow z_4$). Using (2.11)–(2.13), we get, therefore,

$$\sum_{i=1}^{4} C_i = 4I_1(\varepsilon)\, \tilde{Z}(1 \mid 1) + 4f(\delta)\, \tilde{Z}(1 \mid 1) - 4I_1(\varepsilon)\, f(\delta) - 4f^2(\delta) + 4r(\delta) \tag{B.3}$$

Let us consider now the cases of C_5 and C_6. Observe that for the same argument above, $C_5 = C_6$. We now take, for instance, the region $\mathscr{R}^5_\delta = \{z_i \mid |z_1 - z_3| < \delta, |z_2 - z_4| < \delta, |z_2 - z_3|, |z_1 - z_4|, |z_1 - z_2|, |z_3 - z_4| > \delta\}$ and make in $f(1,2,3,4)$ a double expansion of z_3 around z_1 and of z_4 around z_2, namely

$$f(1,2,3,4) = 1 + (z_3 - z_1)^\mu \left.\frac{\partial f}{\partial z_3^\mu}\right|_{\substack{z_3 = z_1 \\ z_4 = z_2}} + (z_4 - z_2)^\mu \left.\frac{\partial f}{\partial z_4^\mu}\right|_{\substack{z_4 = z_2 \\ z_3 = z_1}} + \cdots \tag{B.4}$$

Introducing this in the integrand in C_5, we see that the first term is $D^2(\varepsilon, \delta)$. The first nonvanishing term now is the one containing the fourth derivative

$$\left.\frac{\partial^4 f}{\partial z_3^\mu\, \partial z_3^\nu\, \partial z_4^\alpha\, \partial z_4^\beta}\right|_{\substack{z_3 = z_1\\ z_4 = z_2}}$$

This and all the subsequent terms are finite because the integrands again are either finite or contain integrable singularities for $\beta^2 < 8\pi$ (inside $\mathscr{R}_\delta^5$). We call the sum of all these terms $s(\varepsilon, \delta)$ and of course $s(\delta) = \lim_{\varepsilon \to 0} s(\varepsilon, \delta)$ is finite. Considering (2.11a) and (2.12), we have, therefore,

$$C_5 + C_6 = 2I_1^2(\varepsilon) + 2f^2(\delta) + 4I_1(\varepsilon)\, f(\delta) + 2s(\delta) \tag{B.5}$$

Putting together (B.3) and (B.5), we establish (2.16).

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